CACULATE THE ORIANTATION OF RECTANGLES

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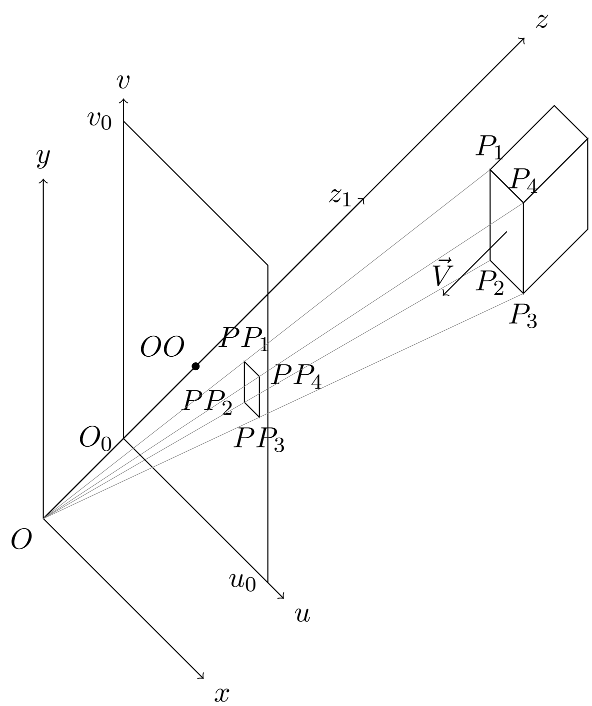
Abstract

This paper shows a method to estimate the orientation of a rectangle in 3D world based exclusively on its 2D projection.

1 notations



As shown in the picture above. O is the optical center of the camera and $\vec{z}_1$ is its optical axis. $(O - xyz)$ is a 3D orthogonal coordinate system and $(O_0 - uv)$ is the image plane with OO as its image center.

The rectangle in 3D world has four points:

$$Points_3 = \begin{pmatrix} \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{P_3} \\ \mathbf{P_4} \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & y_4 \end{pmatrix} \tag{1}$$

Its projection on the image plane also has four points:

$$Points_2 = \begin{pmatrix} \mathbf{PP_1} \\ \mathbf{PP_2} \\ \mathbf{PP_3} \\ \mathbf{PP_4} \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \\ u_4 & v_4 \end{pmatrix} \tag{2}$$

What we want to know is the normal vector of $plane(P_1 - P_4)$:

$$\vec{V} = (x_v \; y_v \; z_v) \tag{3}$$

We are also going to use the notations of some angles:

$$Angles = \begin{pmatrix} \angle A_1 \\ \angle A_2 \\ \angle A_3 \\ \angle A_4 \end{pmatrix} = \begin{pmatrix} \angle P_4 P_1 P_2 \\ \angle P_1 P_2 P_3 \\ \angle P_2 P_3 P_4 \\ \angle P_3 P_4 P_1 \end{pmatrix} \tag{4}$$

2 things that we have already known

2.1 the height and width of the photo(in pixel)

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 720 \\ 1280 \end{pmatrix} \tag{5}$$

2.2 the image center OO which passes $\vec{z}_1$

$$OO = (\frac{u_0}{2} \; \frac{v_0}{2}) \tag{6}$$

2.3 the photo fits the Pinhole Camera Model

for each point $P = (x, y, z)$ and its projection $PP = (u, v)$, it holds true that:

$$\begin{pmatrix} u - \frac{u_0}{2} \\ v - \frac{v_0}{2} \end{pmatrix} = \begin{pmatrix} \frac{x*f}{z} \\ \frac{y*f}{z} \end{pmatrix} \tag{7}$$

where f is the focal length of the camera

2.4 the focal length of the camera (pixel/millimeter)

$$f = 977 \tag{8}$$

2.5 angles of a rectangle are all right angles

$$Angles = \begin{pmatrix} \angle A_1 \\ \angle A_2 \\ \angle A_3 \\ \angle A_4 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \\ \frac{\pi}{2} \\ \frac{\pi}{2} \end{pmatrix} \tag{9}$$

2.6 Claim 1

for any rectangles $Rect^1 = (P_1^1 \; P_2^1 \; P_3^1 \; P_4^1)$ in 3D world, there must exist a rectangle $Rect^2 = (P_1^2 \; P_2^2 \; P_3^2 \; P_4^2)$ sharing the same normal vector $\vec{V_1}$ as well as the same projection on the image plane with $Rect^1$ and it also satisfy that the depth of the first point equal 3000: $P_1^2 = (x_1^2 \; y_1^2 \; 3000.0)$.

this claim ensure us to take any constant as the value of z_1, all of which will lead to the same result of $\vec{V_1}$.

Proof:
let

$$Rect^1 = \begin{pmatrix} x_1^1 & y_1^1 & z_1^1 \\ x_2^1 & y_2^1 & z_2^1 \\ x_3^1 & y_3^1 & z_3^1 \\ x_4^1 & y_4^1 & z_4^1 \end{pmatrix} \tag{10}$$

then

$$Rect^2 = \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 \\ x_2^2 & y_2^2 & z_2^2 \\ x_3^2 & y_3^2 & z_3^2 \\ x_4^2 & y_4^2 & z_4^2 \end{pmatrix} = \begin{pmatrix} \frac{x_1^1*3000}{z_1^1} & \frac{y_1^1*3000}{z_1^1} & 3000 \\ \frac{x_2^1*3000}{z_1^1} & \frac{y_2^1*3000}{z_1^1} & \frac{z_2^1*3000}{z_1^1} \\ \frac{x_3^1*3000}{z_1^1} & \frac{y_3^1*3000}{z_1^1} & \frac{z_3^1*3000}{z_1^1} \\ \frac{x_4^1*3000}{z_1^1} & \frac{y_4^1*3000}{z_1^1} & \frac{z_4^1*3000}{z_1^1} \end{pmatrix} \tag{11}$$

is the rectangle we are seeking.

3 solve the problem

As a result of **claim 1**(2.6), we could simply let $z_1 = 3000.0$ as our first step.
Then we temporarily treat z_2 as a variable.
Once we get the value of z_2, we can calculate $\mathbf{P_1}$ $\mathbf{P_2}$

$$P_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} \frac{u_1-u_0}{f} * z_1 \\ \frac{v_1-v_0}{f} * z_1 \\ z_1 \end{pmatrix} \tag{12}$$

$$P_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{u_2-u_0}{f} * z_2 \\ \frac{v_2-v_0}{f} * z_2 \\ z_2 \end{pmatrix} \tag{13}$$

Then, we can calculate their space vector

$$\overrightarrow{P_1P_2} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix} \tag{14}$$

Next, we calculate the plane $Plane_2$ which is perpendicular to $\overrightarrow{P_1P_2}$ and passes $\mathbf{P_2}$.
(using ax+by+cz=d formed equation)

$$(x_2 - x_1) * x + (y_2 - y_1) * y + (z_2 - z_1) * z = \overrightarrow{P_1P_2} \cdot \mathbf{P_2} \tag{15}$$

For $\mathbf{P_3}$, whose projection is $\mathbf{PP_3}$, it holds true that:

$$\mathbf{P_3} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{u_3-u_0}{f} * z_3 \\ \frac{v_3-v_0}{f} * z_3 \\ z_3 \end{pmatrix} \tag{16}$$

Because $\angle \mathbf{P_1P_2P_3} = \frac{\pi}{2}$, $\mathbf{P_3}$ must lie on $Plane_2$, together with(15)(16)we can calculate the value of z_3:

$$\begin{aligned} (x_2 - x_1) * \frac{u_3 - u_0}{f} * z_3 + (y_2 - y_1) * \frac{v_3 - v_0}{f} * z_3 + (z_2 - z_1) * z_3 = \overrightarrow{P_1P_2} \cdot \mathbf{P_2} \\ \Longrightarrow \\ z_3 = \frac{\overrightarrow{P_1P_2} \cdot \mathbf{P_2}}{(x_2 - x_1) * \frac{u_3-u_0}{f} + (y_2 - y_1) * \frac{v_3-v_0}{f} + (z_2 - z_1)} \end{aligned} \tag{17}$$

According to (16) we can calculate the value of P_3

$$\mathbf{P_3} = \begin{pmatrix} \frac{(u_3-u_0)*\overrightarrow{P_1P_2}\cdot\mathbf{P_2}}{(x_2-x_1)*(u_3-u_0)+(y_2-y_1)*(v_3-v_0)+(z_2-z_1)*f} \\ \frac{(v_3-v_0)*\overrightarrow{P_1P_2}\cdot\mathbf{P_2}}{(x_2-x_1)*(u_3-u_0)+(y_2-y_1)*(v_3-v_0)+(z_2-z_1)*f} \\ \frac{\overrightarrow{P_1P_2}\cdot\mathbf{P_2}}{(x_2-x_1)*\frac{u_3-u_0}{f}+(y_2-y_1)*\frac{v_3-v_0}{f}+(z_2-z_1)} \end{pmatrix} \tag{18}$$

Through the same way as (15)(16)(17)(18), we can calculate the value of P_4

$$\mathbf{P_4} = \begin{pmatrix} \frac{(u_4-u_0)*\overrightarrow{P_1P_2}\cdot\mathbf{P_1}}{(x_2-x_1)*(u_4-u_0)+(y_2-y_1)*(v_4-v_0)+(z_2-z_1)*f} \\ \frac{(v_4-v_0)*\overrightarrow{P_1P_2}\cdot\mathbf{P_1}}{(x_2-x_1)*(u_4-u_0)+(y_2-y_1)*(v_4-v_0)+(z_2-z_1)*f} \\ \frac{\overrightarrow{P_1P_2}\cdot\mathbf{P_1}}{(x_2-x_1)*\frac{u_4-u_0}{f}+(y_2-y_1)*\frac{v_4-v_0}{f}+(z_2-z_1)} \end{pmatrix} \tag{19}$$

Now, we are going to calculate the value of z_2.

Recall that $\angle A_3 = \angle A_4 = \frac{\pi}{2}$, we can write down our object function, and it holds true that the most likely value of z_2 is the value which can make the object function as close to 0 as possible (ideally it should equal 0 but in real world, it can't reach 0 due to the limited resolution of photos.)

$$obj\_func(z_2) = \sqrt[2]{cos^2(\angle A_3) + cos^2(\angle A_4)} \tag{20}$$

where

$$\begin{aligned} cos(\angle A_3) &= \frac{\overrightarrow{P_2P_3} \cdot \overrightarrow{P_3P_4}}{\|\overrightarrow{P_2P_3}\| * \|\overrightarrow{P_3P_4}\|} \\ &= \frac{(x_3 - x_2) * (x_4 - x_3) + (y_3 - y_2) * (y_4 - y_3) + (z_3 - z_2) * (z_4 - z_3)}{\sqrt[2]{(x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2} + \sqrt[2]{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}} \end{aligned} \tag{21}$$

$$
\begin{aligned}
cos(\angle A_4) &= \frac{\overrightarrow{P_3P_4} \cdot \overrightarrow{P_4P_1}}{\|\overrightarrow{P_3P_4}\| * \|\overrightarrow{P_4P_1}\|} \\
&= \frac{(x_4 - x_3) * (x_1 - x_4) + (y_4 - y_3) * (y_1 - y_4) + (z_4 - z_3) * (z_1 - z_4)}{\sqrt[2]{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2} + \sqrt[2]{(x_1 - x_4)^2 + (y_1 - y_4)^2 + (z_1 - z_4)^2}}
\end{aligned}
\tag{22}
$$

Since our object function has only one variable (z_2), there are dozens of methods to find its minimal, I tried particle swarm optimization (PSO), and it worded well.

Once we get the value of z_2, we get the value of $P_1 - P_4$, with which, we can calculate the value of $\vec{V}$ easily.